

# The Non-Trivial Effective Potential of the ‘Trivial’ $\lambda\Phi^4$ Theory: A Lattice Test

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## ABSTRACT

The strong evidence for the ‘triviality’ of  $(\lambda\Phi^4)_4$  theory is not incompatible with spontaneous symmetry breaking. Indeed, for a ‘trivial’ theory the effective potential should be given exactly by the classical potential plus the free-field zero-point energy of the shifted field; i.e., by the one-loop effective potential. When this is renormalized in a simple, but nonperturbative way, one finds, self-consistently, that the shifted field does become non-interacting in the continuum limit. For a classically scale-invariant (CSI)  $\lambda\Phi^4$  theory one finds  $m_h^2 = 8\pi^2 v^2$ , predicting a 2.2 TeV Higgs boson. Here we extend our earlier work in three ways: (i) we discuss the analogy with the hard-sphere Bose gas; (ii) we extend the analysis from the CSI case to the general case; and (iii) we propose a test of the predicted shape of the effective potential that could be tested in a lattice simulation.

# 1 Introduction

The standard model of electroweak interactions is based on the fundamental concept of Spontaneous Symmetry Breaking (SSB) to explain the origin of the vector-boson masses. It is supposed that the complex isodoublet scalar field [1]

$$K(x) = \frac{1}{\sqrt{2}} (\chi_1(x) + i\chi_2(x), v + h(x) + i\chi_3(x)) \quad (1)$$

develops a non-vanishing vacuum expectation value  $v$ . The physical origin of a non-zero  $v$  is, however, hidden in the hitherto-untested part of the theory, namely the “Higgs sector”.

Up to corrections due to the gauge and Yukawa couplings, which are small (assuming that the top mass  $m_t < 200$  GeV), one obtains a simple relation between  $v$  and the Fermi constant  $G_F$ , namely  $v \sim (\sqrt{2}G_F)^{-1/2} \sim 246$  GeV. This estimate represents the phenomenological value of the vacuum field. Thus,  $v$  has to be considered a *renormalized* vacuum expectation value, i.e., one which includes the full dynamical content of the scalar sector, and represents the value of the renormalized scalar field at the minimum of the exact effective potential.

In the presently accepted version of the theory, the explanation for  $v \neq 0$  relies on a semiclassical description of SSB from a  $-\phi^2 + \phi^4$  double-well classical potential with perturbative quantum corrections. In this framework, one has the relation

$$“ m_h^2 = \frac{1}{2} \lambda_R v^2 ” \quad (2)$$

in which  $m_h^2$  is the physical mass of the Higgs particle and  $\lambda_R$  is the renormalized self-coupling of the Higgs field evaluated at external momenta of the order of the Higgs mass itself. On the basis of the above relation, it is generally assumed that a heavy Higgs particle ( $m_h > 0.7$  TeV) is strongly interacting.

However, there is strong evidence that  $(\lambda\Phi^4)_4$  theory is “trivial,” [2, 3, 4, 5] meaning that  $\lambda_R$  vanishes in the continuum limit, which must cast grave doubt on the traditional picture. Various authors [6] claim mass limits around  $m_h < 0.7$  TeV by arguing that “triviality” means that the  $\lambda\Phi^4$  sector of the standard model can only be an effective theory, valid only up to some finite cutoff scale. Without a cutoff, the argument goes, there would be no scalar self-interactions and thus no symmetry breaking [7].

However, “triviality” does not mean that SSB is impossible. The rigorous results do allow a continuum limit of the  $(\lambda\Phi^4)_4$  quantum field theory in which there is a non-zero vacuum expectation value for the field, provided that there are only non-interacting, free-particle excitations above the SSB vacuum. Moreover, as we have argued [8], the

theory can be ‘trivial’ but not ‘entirely trivial’: Although the particles of the theory are non-interacting, the theory can be physically distinguished from a free-field theory: For instance, a phase transition, restoring the symmetry, occurs at a *finite* critical temperature [8, 9]. An analogous ‘trivial’-but-not-entirely-trivial situation occurs in a particular “continuum limit” of the hard-sphere Bose gas, as we discuss in Sect. 2.

In our picture [8], the exact effective potential of massless  $(\lambda\Phi^4)_4$  theory is — because of ‘triviality’ — just the bare classical potential  $\lambda_B\phi_B^4/4!$  plus the zero-point energy of a free-field theory with a mass  $\frac{1}{2}\lambda_B\phi_B^2$  that depends on the constant background field  $\phi_B$ . This object is well known under the name of the ‘one-loop effective potential.’ The natural, nonperturbative renormalization of this effective potential implies that all finite-momentum scattering processes vanish (i.e., ‘triviality’), thus giving a completely self-consistent picture [8].

Exactly the same renormalized effective potential is found, after renormalization, in the Gaussian effective potential (GEP) approach [10, 11]. Originally [11], it was mistakenly assumed that the finding of a non-trivial effective potential had to mean that the theory was interacting. However, it was later realized [12, 13, 14, 15] that there was no conflict with the ‘triviality’ evidence; only the zero-momentum mode of the underlying massless  $\lambda\Phi^4$  theory behaves non-trivially; the finite-momentum modes are non-interacting.

A lattice calculation [16] also finds a non-trivial effective potential, though all lattice studies [4, 5] find that the particle interactions seem to vanish in the continuum limit. As pointed out in Refs. [16, 17], Eq. (2) is completely invalid. The ratio  $m_h^2/v^2$  is not a measure of the Higgs self-coupling strength: it is a finite number, while ‘ $\lambda_R$ ’ vanishes in the continuum limit. If we start with a classically scale-invariant (CSI)  $\lambda\Phi^4$  theory this ratio is  $8\pi^2$ , as shown in Refs. [13, 15, 8]. For  $v = 246$  GeV this predicts a Higgs mass of 2.2 TeV.

In this paper we first discuss the analogy with the non-relativistic Bose gas in Sect. 2. Then we briefly review the main arguments of Refs. [12, 13, 14, 15, 8] in Sects. 3–6. The generalization from the CSI case to include a general bare mass term is discussed in Sect. 7. In Sect. 8 we describe a sharp prediction relating to the shape of the effective potential which could be tested in a high-statistics Monte-Carlo simulation. The conclusions are summarized in Sect. 9.

## 2 Analogy with the hard-sphere Bose gas

The situation of a non-trivial ground state which, however, exhibits non-interacting excitations, can best be understood by analogy with the non-relativistic limit of  $\lambda\Phi^4$ , the “hard-sphere Bose gas” [18]. This model provides an excellent description of the long-wavelength excitations of  $He^4$ , the phonons. The phonon field is just like the Higgs [10]; its creation/annihilation operators are obtained from the original hard-sphere operators  $a(\vec{k})$  and  $a^+(\vec{k})$  after shifting the zero mode and diagonalizing the quadratic Hamiltonian by means of a Bogolubov transformation to new operators  $b(\vec{k})$  and  $b^+(\vec{k})$ . In the text by Huang [18] it is shown that this leads to the effective Hamiltonian: ( $\hbar = 1$ ):

$$H_{\text{eff}} = N \frac{2\pi a}{mv} + \sum_{\vec{k} \neq 0} \frac{k}{2m} \sqrt{k^2 + \frac{16\pi a}{v}} b^+(\vec{k})b(\vec{k}) + \mathcal{O}\left(\frac{a^3}{v}, ak\right). \quad (3)$$

In the above equation  $N$  is the total number of particles,  $v = \frac{V}{N}$  is the average volume per particle,  $a$  the sphere radius. The derivation of  $H_{\text{eff}}$  assumes that the original Bose gas is very *dilute*, i.e.,  $\frac{a^3}{v} \ll 1$ , and also that  $k \ll 1/a$ ; at larger  $k$  there are interactions between the phonons and roton contributions to the spectrum. Note that, for very small  $k$  one has a linear spectrum  $\omega(k) = c_s k$ , where  $c_s \equiv \sqrt{\frac{4\pi a}{m^2 v}}$  is the velocity of sound in  $He^4$ .

Note also that the derivation [18] requires singling out the  $\vec{k} = 0$  mode for special treatment. Bose condensation means that this mode, and only this mode, has a macroscopic occupation number. In fact the depletion,  $D$ , i.e. the fraction of hard-sphere atoms *not* in the  $k = 0$  mode, is small  $D = 1 - \frac{N_0}{N} \sim \frac{a^3}{v} \ll 1$ .

Consider the hypothetical renormalization-group problem of taking the hard-sphere radius  $a$  to zero. In such a limit, the roton branch, starting at momenta  $\sim 1/a$ , is pushed up to infinity; the phonon spectrum becomes exact (by construction) up to arbitrarily high momenta; and phonons have no interactions. If one takes the limit  $a \rightarrow 0$  at fixed density,  $v = \text{const.}$ , then the limit is “entirely trivial” since the effective Hamiltonian reduces to

$$H_{\text{eff}} = \text{const.} + \sum_{\vec{k} \neq 0} \frac{k^2}{2m} b^+(\vec{k})b(\vec{k}), \quad (4)$$

and the Bogolubov matrix is the trivial identity so that  $b(k) = a(k)$  and  $b^+(k) = a^+(k)$ . The speed of sound is now zero, since the gas has infinite compressibility.

However, suppose we take the limit  $a \rightarrow 0$  such that the sound velocity  $c_s$  is kept constant (which corresponds to  $v \sim a$ ). In this situation, the original Bose gas is infinitely dense in physical units ( $\rho/m = 1/v \sim 1/a \rightarrow \infty$ ) but infinitely *dilute* in units of the sphere

volume  $\frac{4\pi}{3}a^3$ , since the ratio between the average distance among the spheres and their radius diverges. In this case the effective Hamiltonian reduces to:

$$H_{\text{eff}} = N\frac{1}{2}mc_s^2 + \sum_{\vec{k} \neq 0} \frac{k}{2m} \sqrt{k^2 + 4m^2c_s^2} b^+(\vec{k})b(\vec{k}). \quad (5)$$

Although no non-trivial S-matrix exists for the phonons in this limit ( $a \rightarrow 0$  with  $c_s = \text{fixed}$ ), their peculiar spectrum, linear at small  $k$ , is quite unlike the trivial spectrum,  $\omega_0(k) = \frac{k^2}{2m}$ . This reveals that the ground state is non-trivial.

The close analogy with relativistic  $\lambda\Phi^4$  can be seen by noting that the observed energy spectrum  $\omega(k)$ , associated with the Bogolubov-transformed operators, can be expressed in terms of the free spectrum  $\omega^{(o)}(k)$  by means of the same universal function [10], namely

$$\omega(k) = \omega^{(o)}(k) \frac{1 + \alpha(k)}{1 - \alpha(k)}, \quad (6)$$

where

$$\alpha(k) = 1 + z - \sqrt{z^2 + 2z}, \quad (7)$$

and  $z = 2\frac{k^2}{B^2}$ ,  $B$  being a characteristic dimensionful scale of the system. In the non-relativistic Bose-gas case,  $\omega^{(o)}(k) = \frac{k^2}{2m}$  and  $B^2 = \frac{16\pi a}{v} = 4m^2c_s^2$ . In the case of the massless relativistic theory one has [10]  $\omega^{(o)} = k$ ,  $B = m_h$  and hence

$$\omega(k) = \sqrt{k^2 + m_h^2}. \quad (8)$$

We shall see in the following section that, just as in the non-relativistic example, all non-trivial dynamical effects of continuum  $\lambda\Phi^4$  can be isolated in the zero mode of the underlying massless theory. This leads to SSB, but with non-interacting particle excitations above the broken-symmetry vacuum. This result, allows one to reconcile the evidence for a non-trivial effective potential with the generally accepted *triviality* of  $(\lambda\Phi^4)_4$ .

### 3 ‘Triviality’ and spontaneous symmetry breaking

Analytical and numerical studies [2, 3, 4, 5] of  $(\lambda\Phi^4)_4$  theory, defined by the Euclidean action

$$\int d^4x \left( \frac{1}{2} \partial_\mu \Phi_B \partial_\mu \Phi_B + \frac{1}{2} m_B^2 \Phi_B^2 + \frac{\lambda_B}{4!} \Phi_B^4 \right), \quad (9)$$

imply that it is a “generalized free field theory.” That is, all renormalized Green’s functions of the continuum theory are expressible in terms of the first two moments of a Gaussian distribution [19]:

$$\tau(x) = v, \quad (10)$$

$$\tau(x, y) = v^2 + G(x - y), \quad (11)$$

so that

$$\tau(x, y, z) = v^3 + v(G(x - y) + G(x - z) + G(y - z)), \quad (12)$$

$$\tau(x, y, z, w) = v^4 + v^2(G(x - y) + \text{perm.}) + G(x - y)G(z - w) + \text{perm.}, \quad (13)$$

and so on. Here,  $v$  is a constant (since we assume that translational invariance is not broken), and  $G(x - y)$  is just a free propagator with some mass  $m_h$ . Moreover, it has residue  $Z_h = 1$ , since it must satisfy a Källén-Lehmann representation with a spectral function  $\delta(s - m_h^2)$ . The index “ $h$ ” in  $Z_h$  and  $m_h$  refers to the shifted field  $h(x)$  introduced by means of a suitably de-singularized, renormalized field operator  $\Phi_R(x)$ , such that  $\langle \Phi_R(x) \rangle = v$  and  $h(x) \equiv \Phi_R(x) - v$ . The above equations imply that all *connected* three- and higher-point Green’s functions of the  $h(x)$  field vanish; *i.e.*, ‘triviality’.

By its very nature, the generalized free-field structure dictates a trivially free shifted field, but it does not forbid a non-zero value of  $v$  [5]. Thus, it should be possible for the theory to have a non-trivial effective potential  $V_{\text{eff}}$  with SSB minima. This is precisely what is found in one-loop, Gaussian, and lattice calculations [16, 17], all of which are completely consistent with ‘triviality’ for the shifted field.

For the effective potential to be non-trivial the zero-momentum mode of the underlying theory must behave non-trivially. This immediately suggests that one should concentrate on *massless*  $\lambda\Phi^4$  theories, for which zero-momentum ( $p_\mu = 0$ ) represents a physical, on-shell point. The ground state of a free, massless scalar theory is infinitely degenerate — the potential is zero — and Bose-Einstein condensation occurs for zero coupling. Therefore, the perturbative ground state is *essentially* unstable, even for vanishingly small coupling.

Since the massless theory contains no intrinsic scale, the physical scale,  $v$  (with  $m_h$  proportional to  $v$ ), must be spontaneously generated by “dimensional transmutation.” This is exactly the philosophy of Coleman and Weinberg [20]. “Dimensional transmutation” requires the existence of a non-trivial Callan-Symanzik  $\beta$  function. Usually one would obtain the  $\beta$  function perturbatively from the momentum dependence of the 4-point function at finite momentum. However, in  $(\lambda\Phi^4)_4$  theory that approach is doomed to failure since ‘triviality’ means that any such ‘renormalized coupling constant’ must vanish. To extract a more meaningful  $\beta$  function one must start from a quantity that will be finite, and *non-vanishing* in the infinite-cutoff limit.  $V_{\text{eff}}$  is such a non-trivial quantity and one can extract from it a nonperturbatively defined  $\beta$  function which is negative. This implies that the *bare* coupling constant must go to zero as the ultraviolet regulator is removed. This corresponds to the delicate case in the rigorous analyses [3]. (See also Ref. [21].)

However, it is perhaps best to avoid the phrase “asymptotic freedom” for this property ( $\lambda_B \rightarrow 0$  in the continuum limit) because it has nothing to do with the existence of a renormalized coupling ‘ $\lambda_R(Q^2)$ ’ which decreases to zero as  $Q^2$  increases.

## 4 The effective potential

Consider the action (9) in the CSI case. Making a shift of the field,  $\Phi_B(x) = \phi_B + h(x)$  (requiring  $\int d^4x h(x) = 0$  to avoid ambiguity), one finds  $h^2, h^3, h^4$  terms. Ignoring the ‘bare interaction’ terms,  $h^3, h^4$ , one has a free  $h(x)$  field with a  $\phi_B$ -dependent mass-squared;  $\frac{1}{2}\lambda_B\phi_B^2$ , in the CSI case. The corresponding effective potential for  $\phi_B$  is just the classical potential plus the zero-point energy of the  $h(x)$  field:

$$V_{\text{eff}} = \frac{\lambda_B}{4!}\phi_B^4 + \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \ln(p^2 + \frac{1}{2}\lambda_B\phi_B^2), \quad (14)$$

which is the so-called one-loop effective potential [20, 22]. In our picture, this will effectively give the exact result, with all effects of the ‘bare interactions’ being re-absorbable into the renormalized parameters.

After subtracting a constant and performing the mass renormalization so that the second derivative of the potential vanishes at the origin, one has [20]:

$$V_{\text{eff}} = \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda_B^2\phi_B^4}{256\pi^2} \left( \ln \frac{\frac{1}{2}\lambda_B\phi_B^2}{\Lambda^2} - \frac{1}{2} \right), \quad (15)$$

where  $\Lambda$  is an ultraviolet cutoff. This function, being just a sum of  $\phi_B^4 \ln \phi_B^2$  and  $\phi_B^4$  terms, necessarily has a pair of minima at some value  $\pm v_B$ . It may therefore be re-written in the form:

$$V_{\text{eff}} = \frac{\lambda_B^2\phi_B^4}{256\pi^2} \left( \ln \frac{\phi_B^2}{v_B^2} - \frac{1}{2} \right). \quad (16)$$

Comparing the equivalent forms (15) and (16) gives  $v_B$  in terms of  $\Lambda$ . Hence, one finds for the particle mass in the SSB vacuum:

$$m_h^2 = \frac{1}{2}\lambda_B v_B^2 = \Lambda^2 \exp \left( -\frac{32\pi^2}{3\lambda_B} \right). \quad (17)$$

Demanding that the particle mass be finite, one thus finds an infinitesimal  $\lambda_B$ :

$$\lambda_B = \frac{32\pi^2}{3} \frac{1}{\ln(\Lambda^2/m_h^2)}. \quad (18)$$

The effective potential can be made manifestly finite by re-scaling the constant background field  $\phi_B$ . That is, one can define a renormalized  $\phi_R$  as  $Z_\phi^{-1/2}\phi_B$ , where  $Z_\phi$  must

go to infinity as  $\ln(\Lambda^2/m_h^2)$ , so that  $\lambda_B Z_\phi$  is finite, and hence  $m_h^2$  is finitely proportional to  $v \equiv v_R$ . The absolute normalization of  $Z_\phi$  is fixed by requiring the second derivative of  $V_{\text{eff}}$  with respect to  $\phi_R$  at  $\phi_R = v$  to agree with  $m_h^2$ , as discussed in the next section. Thus, one obtains:

$$V_{\text{eff}} = \pi^2 \phi_R^4 \left( \ln \frac{\phi_R^2}{v^2} - \frac{1}{2} \right), \quad (19)$$

and

$$m_h^2 = 8\pi^2 v^2. \quad (20)$$

(This is for the CSI case; see Sect. 6 for the results for the general form of  $m_B$ .) These results should be considered *exact* if the ‘triviality’ structure (10–13) is exact [8]. [To be pedantic, the exact effective potential is the ‘convex envelope’ of our  $V_{\text{eff}}$ ; see Refs. [23]–[27].]

## 5 The field renormalization

Just as in the non-relativistic case, a proper quantization of the massless theory requires special treatment of the zero mode (which is essentially a classical object [26]). Therefore, the crucial initial step in the above calculation was to separate the full, bare quantum field as

$$\Phi_B(x) = \phi_B + h_B(x). \quad (21)$$

Recall that, to avoid ambiguity,  $h_B(x)$  is required to satisfy  $\int d^4x h_B(x) = 0$ ; this means that it has no Fourier projection onto the  $p_\mu = 0$  mode. This decomposition is Lorentz invariant, of course. Thus, in principle, one disposes of two renormalization constants  $Z_\phi$  and  $Z_h$ , with  $\phi_B^2 = Z_\phi \phi_R^2$ , and  $h_B^2(x) = Z_h h_R^2(x)$ .  $Z_h$ , as usual, has to be determined from the *variation* of the self-energy with  $p^2$ , and has to approach  $Z_h = 1$  in the continuum limit to reproduce Eqs. (10–13). However,  $Z_\phi$ , which concerns the constant field with no projection out of  $p_\mu = 0$ , is related to the renormalization-group properties of the effective potential. The RG analysis requires that  $Z_\phi$  is infinite, of order  $\ln(\Lambda^2/m_h^2)$ , so that  $\lambda_B \phi_B^2$  is finitely proportional to  $\phi_R^2$ .

It is crucial to our picture that the  $Z_\phi^{1/2}$  re-scaling of the constant background field  $\phi_B$  is quite distinct from the  $Z_h^{1/2} = 1$  re-scaling of the fluctuation field  $h(x)$ . This structure is more general than in perturbation theory, and is the basic ingredient [12, 13, 14, 15, 8] that allows one to understand how non-trivial SSB co-exists with a ‘trivial’ non-interacting shifted field. The interactions of the  $h(x)$  field go to zero because  $\lambda_B$  vanishes, but the effective potential remains non-trivial because  $\lambda_B \rightarrow 0$  is compensated by  $Z_\phi \rightarrow \infty$ .



In Ref. [8] it is shown that the separate  $\phi$  and  $h$  re-scalings can, in fact, be expressed as a single, overall re-scaling of the whole field, provided that one uses a momentum-dependent  $Z^{1/2}(p)$ :

$$Z^{\frac{1}{2}}(p) = Z_{\phi}^{\frac{1}{2}}\mathcal{P} + Z_h^{\frac{1}{2}}\overline{\mathcal{P}}, \quad (22)$$

where

$$\mathcal{P} \equiv \frac{\bar{\delta}^4(p)}{\bar{\delta}^4(0)} \quad \text{and} \quad \overline{\mathcal{P}} = 1 - \mathcal{P} \quad (23)$$

are orthogonal projections ( $\mathcal{P}^2 = \mathcal{P}$ ,  $\overline{\mathcal{P}}^2 = \overline{\mathcal{P}}$ ,  $\mathcal{P}\overline{\mathcal{P}} = 0$ ) which select and remove the  $p^\mu = 0$  mode, respectively. [Here  $\bar{\delta}^4(p) \equiv (2\pi)^4\delta^4(p)$ , and  $\bar{\delta}^4(0)$  has the usual interpretation as the spacetime volume.]

$V_{\text{eff}}$  is the generator of the zero-momentum Green's functions:

$$V_{\text{eff}}(\phi_B) = V_{\text{eff}}(v_B) - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma_B^{(n)}(0, 0, \dots; v_B) (\phi_B - v_B)^n \quad (24)$$

$$= V_{\text{eff}}(v_R) - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma_R^{(n)}(0, 0, \dots; v_R) (\phi_R - v_R)^n, \quad (25)$$

where

$$\Gamma_R^{(n)}(0, 0, \dots; v_R) = Z_{\phi}^{n/2} \Gamma_B^{(n)}(0, 0, \dots; v_B). \quad (26)$$

(Recall that  $V_{\text{eff}}(\phi_B) = V_{\text{eff}}(\phi_R)$ , the effective potential being a renormalization-group-invariant quantity.) The  $\Gamma_R^{(n)}$ 's at *zero* momentum, being derivatives of the renormalized effective potential, are finite. However, at finite momentum, the  $\Gamma_R^{(n)}$ 's should vanish for  $n \geq 3$ , corresponding to 'triviality'. Thus, the  $p^\mu \rightarrow 0$  limit is not smooth; the zero mode has non-trivial interactions, but the finite-momentum modes do not. However, the 2-point function at finite momentum is  $\Gamma_R^{(2)}(p) = p^2 + m_h^2$ , which is the (Euclidean) inverse propagator of a free field of mass  $m_h$ . This will have a smooth limit at  $p^\mu = 0$ , provided we require

$$\left. \frac{d^2 V_{\text{eff}}(\phi_R)}{d\phi_R^2} \right|_{\phi_R=v_R} = m_h^2. \quad (27)$$

This condition fixes the absolute normalization of  $Z_{\phi}$ . The point is this: The  $h(x)$ -field fluctuations (which in some sense are infinitesimal on the scale of  $\phi_R$  if they were finite on the scale of  $\phi_B$ ) are only sensitive to the quadratic dependence of  $V_{\text{eff}}$  in the neighbourhood of  $v_R$ . This quadratic dependence should correspond, self consistently, to the potential for a free field of mass  $m_h$ .

To conclude this section, we stress that, as pointed out in the introduction, the value  $v$  entering Eq. (1), the expression for the isodoublet scalar field in the Weinberg-Salam

model, has to be considered a cutoff independent, renormalized quantity. Thus, the field  $K(x)$  in Eq. (1) is simply the  $O(4)$  extension of our *renormalized* field  $\Phi_R(x) = \phi_R + h(x)$  evaluated at the minimum  $\phi_R = v_R \equiv v$ . (We may write  $h(x) = h_R(x) = h_B(x)$  since  $Z_h = 1$ .) The basic phenomenological consequences of this identification are discussed briefly in Sect. 9, and in more detail in [13, 15, 8].

## 6 Appropriate and inappropriate methods for calculating the effective potential in $\lambda\Phi^4$ theory

If  $(\lambda\Phi^4)_4$  theory is indeed ‘trivial,’ as we believe, then one must be careful about what methods one uses to compute the effective potential. Spurious contradictions will inevitably arise if one tries to use an approximation method that is inherently incompatible with the ‘triviality’ structure (10–13). Thus, perturbation theory, the loop expansion (beyond one loop), and leading-log re-summation are all wholly misleading because they insist upon having a finite connected 4-point function at non-zero external momenta.

‘Triviality’ implies that the effects of the bare  $h$ -field interactions, in total, produce no observable particle interactions. One may either ignore the bare interactions entirely, or re-sum some consistent subset of their effects. What is disastrous, though, is to take into account only some of the bare interactions in a perturbative or quasi-perturbative manner.

The only known approximations to the effective potential which are compatible with the generalized free-field structure (10–13) are the one-loop and the Gaussian approximations [28, 29, 10, 11, 30]. In the first case the self-interaction effects of the shifted field are consistently neglected, while in the Gaussian approximation a consistent infinite subset of bare self-interactions are re-summed. As discussed in detail in Refs. [8, 13, 14, 15], both approximations yield exactly the same renormalized results for the effective potential and for the ratio of  $m_h^2$  to the renormalized vacuum value  $v$ , namely Eqs. (19, 20).

It is possible, in principle, to consider other approximations to the effective potential that “improve” upon the one-loop or Gaussian approximation, in that they take into account a larger subset of the bare  $h$  interactions. However, such approximations must be compatible with the possibility that there are no observable  $h$ -particle interactions. For example, one could consider post-Gaussian variational calculations (either Hamiltonian [31] or covariant [32]) in the spirit of the effective potential for composite operators introduced by Cornwall, Jackiw, and Tomboulis (CJT) [33]. CJT show that there is an exact

relation:

$$\int d^3x V_{\text{eff}}(\phi) = E[\phi, G_o(\phi)], \quad (28)$$

where  $E[\phi, G]$  is  $\min \langle \Psi | H | \Psi \rangle$ , minimized over all normalized states  $|\Psi\rangle$ , subject to the conditions  $\langle \Psi | \Phi | \Psi \rangle = \phi$  and  $\langle \Psi | \Phi(\vec{x}, t) \Phi(\vec{y}, t) | \Psi \rangle = \phi^2 + G(\vec{x}, \vec{y})$ , and the full propagator,  $G_o(\phi)$ , is obtained from

$$\left. \frac{\delta E}{\delta G(\vec{x}, \vec{y})} \right|_{G=G_o(\phi)} = 0. \quad (29)$$

A consistent approximation, in our sense, is one in which this variational structure is properly respected. That is, for a given approximate  $E[\phi, G]$ , one must solve Eq. (29) exactly. To solve this equation only in a quasi-perturbative manner will lead to inconsistencies. However, in a consistent calculation — no matter how sophisticated the approximation to  $E[\phi, G]$  is — we would expect the optimal  $G$  to reduce to a free propagator, and our equations (19, 20) to remain unmodified in the continuum limit.

In other words, the ‘triviality’ of  $(\lambda\Phi^4)_4$  theory implies that the bare  $h^3, h^4$  interaction term is an “irrelevant” operator, in the sense that, in a *consistent* approximation to the effective potential, i.e., compatible with Eqs. (10–13) in the continuum limit, all  $h$ -field bare self-interaction effects are re-absorbable in the renormalization process, leaving the physically relevant relations (19, 20) unchanged.

## 7 General, non-classically-scale-invariant, case

In Ref. [8] we considered only the classically scale-invariant (CSI)  $\lambda\Phi^4$  theory, characterized by a single parameter  $v$ , the scale produced by dimensional transmutation. In this section we discuss briefly the general case which involves a second parameter  $m_0$ . The CSI case corresponds to *exactly* zero mass for the particles of the symmetric phase, and in dimensional regularization, or any such regularization in which scale-less quadratic-divergent integrals are set to zero, it corresponds simply to  $m_B = 0$ . However, we try to avoid calling this “the massless case” because, in our picture, the only not-entirely-trivial  $\lambda\Phi^4$  theories have massless particles in the symmetric phase. That is, even in the general case  $m_B^2$  has to be infinitesimally close to the CSI form, so that the particles of the symmetric phase always have vanishingly small mass in the continuum limit. If  $m_B$  differs finitely from the CSI form in the continuum limit then one is too far away from the phase transition and will obtain only an entirely trivial theory. (Either the theory is in a trivial, massive, symmetric phase, or it is so far into the broken phase that the symmetry cannot

be restored at any finite temperature: both cases are physically indistinguishable from a massive free field theory.)

We view the CSI case ( $m_0 = 0$ ) as by far the most attractive theoretical possibility, for the same aesthetic reasons as Coleman and Weinberg [20]: The classical  $\lambda\Phi^4$  action — and thus the whole Standard Model action — then contains no dimensionful parameter. The physically observed scale is then purely a consequence of the quantum anomaly that leads to “dimensional transmutation.” Given the increasing theoretical evidence that scale and conformal invariance play a very deep role in physics, we are convinced that the CSI case is the one that Nature has chosen.

However, the general case is worth considering to gain a fuller understanding, and in order to compare with lattice and other calculations. The Gaussian-effective-potential (GEP) analysis of Ref. [11] (see also [34, 35]) treats the general case, and a parallel analysis can be done in the one-loop context [36]. Here we follow the GEP analysis [11], but incorporating the proper normalization of the renormalized constant field, determined by Eq. (27) [37].

The GEP is obtained by a variational calculation using a variational parameter  $\Omega$ . Expressing the field  $\Phi_B(x)$  as  $\phi_B + h(x)$ , one first computes  $V_G(\phi_B, \Omega)$ , which is the expectation value of the Hamiltonian in a trial state  $|0\rangle_\Omega$ , which is a free-field vacuum state with mass  $\Omega$  for the  $h(x)$  field. A straightforward computation yields [29]:

$$V_G(\phi_B, \Omega) = I_1(\Omega) + \frac{1}{2}(m_B^2 - \Omega^2)I_0(\Omega) + \frac{1}{2}m_B^2\phi_B^2 + \frac{\lambda_B}{4!} \left( \phi_B^4 + 6I_0(\Omega)\phi_B^2 + 3I_0^2(\Omega) \right), \quad (30)$$

where

$$I_n(\Omega) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} (\omega_k^2)^n, \quad \omega_k^2 \equiv \vec{k}^2 + \Omega^2. \quad (31)$$

The integral  $I_1(\Omega)$  represents the zero-point energy for a free field of mass  $\Omega$ , and  $I_0(\Omega)$  is  $\langle h(x)^2 \rangle_\Omega$ . Minimizing with respect to the variational parameter  $\Omega$  yields an equation determining the optimum  $\Omega$  as a function of  $\phi_B$ :

$$\Omega^2 = m_B^2 + \frac{1}{2}\lambda_B(I_0(\Omega) + \phi_B^2). \quad (32)$$

The GEP,  $\bar{V}_G(\phi_B)$ , results when  $V_G(\phi_B, \Omega)$  is evaluated using this optimum  $\Omega$ . It is convenient to note that the first derivative of the GEP can be simply expressed as:

$$\frac{1}{2\phi_B} \frac{d\bar{V}_G}{d\phi_B} = \frac{d\bar{V}_G}{d(\phi_B^2)} = \frac{1}{2}(\Omega^2 - \frac{1}{3}\lambda_B\phi_B^2). \quad (33)$$

In the general case the mass renormalization takes the form [11]:

$$m_B^2 = -\frac{1}{2}\lambda_B I_0(0) + \frac{m_0^2}{8\pi^2 I_{-1}(\mu)}, \quad (34)$$

where  $I_{-1}(\mu)$ , from Eq. (31), is a log-divergent integral. (It corresponds to  $1/(4\pi^2\epsilon)$  in dimensional regularization, or to  $(1/8\pi^2)\ln(\Lambda^2/\mu^2)$  with an ultraviolet cutoff  $\Lambda$ .) The first term in  $m_B^2$  serves to cancel the quadratic divergences of the theory (in dimensional regularization it can be consistently set to zero). The second term, which introduces the finite parameter  $m_0^2$ , is infinitesimal, and it must be so if  $V_{\text{eff}}$  is to be finite. If one tried to include a finite term in  $m_B^2$  one would obtain only an entirely trivial theory. A systematic derivation of the above form of  $m_B^2$  can be given by generalizing the RG procedure used in Refs. [30, 8]; see Ref [27].

Substituting  $m_B^2$  into the  $\Omega$  equation and using the formula [34, 29]:

$$I_0(\Omega) = I_0(0) - \frac{1}{2}\Omega^2 I_{-1}(\mu) + g(\Omega), \quad (35)$$

with

$$g(\Omega) \equiv \frac{\Omega^2}{16\pi^2} \left( \ln \frac{\Omega^2}{\mu^2} - 1 \right), \quad (36)$$

one sees that the quadratic divergences cancel. The renormalization proceeds as in the CSI case: we need an infinitesimal  $\lambda_B$  of the form:

$$\lambda_B = 2/I_{-1}(\mu), \quad (37)$$

and an infinite re-scaling of the constant field,  $\phi_B^2 = Z_\phi \phi_R^2$ , with

$$Z_\phi = 12\pi^2 \zeta I_{-1}(\mu). \quad (38)$$

The factor  $\zeta$  is to be fixed by imposing the condition (27); it will depend on  $m_0^2$ , and the  $12\pi^2$  has been included so that  $\zeta = 1$  in the CSI case. The  $\Omega$  equation then reduces to:

$$\Omega^2 = 8\pi^2 \zeta \phi_R^2 + \frac{2}{3} \left( g(\Omega) + \frac{m_0^2}{8\pi^2} \right) \frac{1}{I_{-1}(\mu)}. \quad (39)$$

Thus,  $\Omega^2$  is finitely proportional to  $\phi_R^2$ , up to infinitesimal terms, for any  $m_0$ . It would be wrong to call  $m_0$  “the renormalized mass,” since it is *not* the particle mass in the symmetric phase; it is just a finite parameter with dimensions of mass. In the continuum limit the particle mass in the symmetric phase vanishes for any  $m_0$ .

However, the extra  $m_0^2$  term in  $m_B^2$  does produces an extra term in  $\bar{V}_G$ , since

$$\begin{aligned} \frac{d\bar{V}_G}{d(\phi_R^2)} &= 12\pi^2 \zeta I_{-1}(\mu) \frac{d\bar{V}_G}{d(\phi_B^2)} \\ &= 12\pi^2 \zeta I_{-1}(\mu) \frac{1}{2} (\Omega^2 - 8\pi^2 \zeta \phi_R^2) \end{aligned}$$

$$\begin{aligned}
&= 4\pi^2\zeta \left( g(\Omega) + \frac{m_0^2}{8\pi^2} \right) \\
&= 2\pi^2\zeta^2\phi_R^2 \left( \ln \frac{8\pi^2\zeta\phi_R^2}{\mu^2} - 1 \right) + \frac{1}{2}m_0^2\zeta,
\end{aligned} \tag{40}$$

where, in the last step we use (39), and discard the  $\mathcal{O}(1/I_{-1})$  terms. Integrating the last equation with respect to  $\phi_R^2$  we obtain  $\bar{V}_G$  in renormalized form. It contains an  $m_0^2\phi_R^2$  term in addition to the  $\phi^4 \ln \phi_R^2$  and  $\phi_R^4$  terms of the CSI case. Eliminating  $\mu$  in favour of the vacuum value  $v$ , it can be conveniently written in the form:

$$V_{\text{eff}}(\phi_R) = \pi^2\zeta^2\phi_R^4 \left( \ln \frac{\phi_R^2}{v^2} - \frac{1}{2} \right) + \frac{1}{2}m_0^2\zeta\phi_R^2 \left( 1 - \frac{1}{2} \frac{\phi_R^2}{v^2} \right). \tag{41}$$

(It is easily verified that the derivative vanishes at  $\phi_R = v$ , as required.)

Next, we impose the consistency condition (27) that the second derivative of the effective potential at  $v$  should agree with the physical mass of the SSB vacuum,  $m_h^2 \equiv \Omega^2(\phi_R = v)$ . This determines  $\zeta$  to be

$$\zeta = 1 + \frac{m_0^2}{4\pi^2v^2}. \tag{42}$$

Therefore, from (39) at  $\phi_R = v$ , the physical mass is

$$m_h^2 = 8\pi^2\zeta v^2 = 8\pi^2v^2 + 2m_0^2. \tag{43}$$

Note that one may use Eq. (42) to eliminate  $m_0^2$  for  $\zeta$  (or *vice versa*) in Eq. (41). One can easily check that  $\phi_R = v$  is a *minimum* of the potential for all  $\zeta > 0$ . This minimum has a lower energy than the origin if  $\zeta < 2$ . Thus, the situation is this: for  $\zeta > 2$  the symmetric vacuum is stable; at  $\zeta = 2$  there is a phase transition to the broken-symmetry phase; as  $\zeta$  is decreased one gets deeper into the broken phase. At  $\zeta = 1$  one reaches the CSI case, and in the limit  $\zeta \rightarrow 0$  one has the ‘extreme double-well’ limit where the shape of the effective potential approaches the classical quartic-polynomial form.

For sufficiently large  $m_0^2$ , such that  $\zeta > 2$ , it is possible to have the symmetric phase be stable. This would contain *massless* particles which would behave non-trivially. Scattering amplitudes would be singular for any fixed number of particles, due to infrared divergences, but there should be sensible dynamics for suitably defined coherent states containing an indefinite number of particles. This would correspond to the non-trivial massless  $(\lambda\Phi^4)_4$  theory constructed by Pedersen, Segal, and Zhou [38].

## 8 A possible lattice test

The shape of the effective potential associated with ‘triviality’ is a definite prediction which can be tested in a computer simulation along the lines of the calculation of Huang, Manousakis, and Polonyi (HMP) [16]. The test we propose here requires calculation only of the effective potential, and not of the propagator or higher-point functions, and is independent of any re-scaling of the constant  $\phi$  field.

One starts with the bare Euclidean action (9) expressed in a discretized, lattice form. The ultraviolet cutoff  $\Lambda$  can basically be identified with  $\pi/a$ , where  $a$  is the lattice spacing. One should keep the bare coupling  $\lambda_B$  at values of order unity or smaller, so that  $\frac{\lambda_B}{16\pi^2} \ll 1$ , and hence  $m_h^2 \ll \Lambda^2$  (see Eq. (17)). One then couples the system to an external, constant source  $J$ , and runs a simulation to calculate the average bare field

$$\phi_B = \langle \Phi_B \rangle_J \quad (44)$$

as a function of  $J$  and  $m_B^2$ . Inverting this relation gives  $J$  as function of  $\phi_B$  and  $m_B^2$ . But, by the usual Legendre-transform property,  $J$  is just the derivative of the effective potential:

$$\frac{dV_{\text{eff}}}{d\phi_B} = J = J(\phi_B, m_B^2). \quad (45)$$

Thus, the lattice data can be compared with our predicted form of the continuum limit of  $V_{\text{eff}}$ .

From Eq. (41), using (42), and then re-expressing the result back in terms of the bare field, the predicted form is:

$$J = \frac{1}{Z_\phi^{1/2}} \frac{dV_{\text{eff}}}{d\phi_R} = \frac{4\pi^2 \zeta^2}{Z_\phi^2} \phi_B \left[ \phi_B^2 \ln \frac{\phi_B^2}{v_B^2} + \frac{(\zeta - 1)}{\zeta} (v_B^2 - \phi_B^2) \right]. \quad (46)$$

Only the overall coefficient is sensitive to the field re-scaling. Recall that  $\zeta$  is 1 in the CSI case and  $\zeta$  is 2 at the phase transition.

Ideally, one would like to make the comparison at the value of  $m_B$  that corresponds to the CSI case ( $\zeta = 1$ ). However, it is not quite clear how to identify this case on the lattice. To avoid this problem one can make the comparison precisely at the phase transition,  $\zeta = 2$ . On the lattice this means at  $m_B^2 = m_c^2(\lambda_B)$ , where, for  $m_B^2 > m_c^2$  the only solution of  $J(\phi_B, m_B^2) = 0$  is at  $\phi_B = 0$ , while for  $m_B^2 \leq m_c^2$  that is not true.

To illustrate the point, consider the ratio  $B^2/(AC)$ , where  $A$ ,  $B$ , and  $C$  are the first, second, and third derivatives, respectively, of  $J$  at the vacuum:

$$J(\phi_B) = A(\phi_B - v_B) + \frac{B}{2!}(\phi_B - v_B)^2 + \frac{C}{3!}(\phi_B - v_B)^3 + \dots \quad (47)$$

These coefficients must be evaluated from data in the region  $|\phi_B| > v_B$ , since  $J$  is zero in the region  $-v_B < \phi_B < v_B$ , reflecting the convexity of the effective potential [23]: see Fig. 1. The  $B^2/(AC)$  ratio is completely independent of any re-scaling of  $\phi_B$ . From our formula (46) we find:

$$\frac{B^2}{AC} = \frac{(3 + 2\zeta)^2}{(3 + 8\zeta)} \quad (48)$$

At the phase transition ( $\zeta = 2$ ) this is  $49/19 = 2.579$ . In the CSI case ( $\zeta = 1$ ) it would be  $25/11 = 2.273$ , and the smallest allowed value is  $9/4 = 2.25$ , occurring at  $\zeta = 3/4$ . These may be compared with the result for a classical  $\phi^2(\phi^2 - 2v^2)$  potential, which is 3. This corresponds to the limit  $\zeta \rightarrow 0$ .

The predicted ratio at the phase transition,  $49/19 = 2.579$ , could be tested in a high-statistics Monte-Carlo simulation. Notice that, this test does not require calculating the irreducible two-point function in the broken phase. Obviously, further tests become possible if the physical mass is also calculated. (For instance, the physical mass at the phase transition is  $16\pi^2 v^2$ , so from it and  $v_B$  one can infer  $Z_\phi$ , which can then be checked against the overall factor in Eq. (46) for  $\zeta = 2$ .)

Deviations of the ratio  $B^2/(AC)$  from our predicted value represent deviations from ‘triviality’: They represent a measure of the residual self-interaction effects of the shifted field which are not absorbed in renormalization. In our picture they must vanish, though only slowly, as an inverse power of  $\ln \Lambda$ , in the continuum limit. Assuming that a lattice calculation can approach sufficiently close to the continuum limit in the appropriate range of  $\lambda_B, m_B^2$ , one can then explicitly test the effective-potential shape associated with ‘triviality’.

An analysis of the published data of HMP [16], discussed in the Appendix, seems to be consistent with our picture, although much greater precision and closer approach to the continuum limit is needed for a real test.

## 9 Conclusions

‘Triviality’ can naturally co-exist with non-trivial SSB. The effective potential is then just the classical potential plus the zero-point energy of the effectively-free shifted field. The SSB is non-trivial in the sense that the symmetry can be restored at a finite critical temperature [8, 9]. Thus, the theory is not entirely trivial; it can be *physically* distinguished from a free-field theory. This situation has a simple analog in the hard-sphere Bose gas (Sect. 2.).



In this picture the one-loop effective potential becomes effectively exact, and this is verified by the fact that the same result is found, after renormalization, in the Gaussian approximation. The nonperturbative renormalization leads, self-consistently, to the conclusion that the shifted field's interactions are infinitely suppressed.

In the general case (Sect. 7) the renormalized theory is characterized by two parameters  $v$  and  $m_0$  (or  $v$  and  $\zeta \equiv 1 + m_0^2/(4\pi^2 v^2)$ ) that replace  $\lambda_B$  and  $m_B^2$ . However, the most theoretically attractive case is when  $m_0 = 0$ , since the theory is then classically scale invariant. In this case  $m_h^2 = 8\pi^2 v^2$ . Since, phenomenologically,  $v$  is 246 GeV, this predicts a 2.2 TeV Higgs boson.

It is usually assumed that such a heavy Higgs must be strongly interacting and be a very broad resonance. However, this assumption is based on the naive classical formula (2) that has “ $\lambda_R$ ,” a measure of the scalar-sector interaction strength, proportional to  $m_h^2/v^2$ . However, that is inconsistent with ‘triviality’, which says that “ $\lambda_R$ ” should be infinitesimally small.

In our picture, the Higgs, although very heavy, is weakly interacting, as are the longitudinal gauge bosons. Indeed, the scalar sector would be completely non-interacting were it not for the gauge couplings  $g, g'$ . Although the scalar sector must be treated non-perturbatively, one may continue to treat the gauge interactions using perturbation theory. Effectively, then, inclusive electroweak processes can be computed as usual, provided one uses a renormalizable gauge and sets the Higgs self-coupling and its coupling to the Higgs-Kibble ghosts (the would-be Goldstones) to zero. One should avoid the so-called ‘unitary gauge’ and the naive use of  $W, Z$  polarization vectors [39].

For instance, consider the Higgs decay width to  $W$  and  $Z$  bosons. The conventional calculation would give a huge width, of order  $G_F m_h^3 \sim m_h$ . However, in a renormalizable-gauge calculation of the imaginary part of the Higgs self-energy, this result comes from a diagram in which the Higgs supposedly couples strongly to a loop of Higgs-Kibble ghosts. That diagram is effectively absent in our picture, leaving a width of order  $g^2 m_h$ . Thus, in our picture the Higgs is a relatively narrow resonance, decaying predominantly to  $t\bar{t}$  quarks.

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## Appendix: Analysis of existing lattice data

The published data of HMP [16, 17] for  $J$  as a function of  $\phi_B$  and  $m_B^2$  already allows a rough test of our predicted form of the effective potential. The data were collected in 1987, running on a VAX, with a  $10^4$  lattice. A simulation with greater precision on a larger lattice should be perfectly feasible, and is really needed for a meaningful test of the validity of our picture.

HMP's Fig. 2 gives results for ' $\lambda_0 = 1$ ', which corresponds to our  $\lambda_B = 6$ . The phase transition is near  $m_B^2 = -0.4$  in lattice units, but unfortunately this is just *before* the transition. We are forced to go to the next value,  $m_B^2 = -0.6$ , where  $v_B$  is  $0.436 \pm 0.004$ . The pairs  $(J, \phi_B)$  for this case, extracted from HMP's figure, are tabulated in Table 1.

We start with a model-independent 3-parameter fit using the form of  $J$  quoted in Eq. (47). This gives

$$A = 0.31 \pm 0.03, \quad (A1)$$

$$B = 2.15 \pm 0.30, \quad (A2)$$

$$C = 6.41 \pm 1.14, \quad (A3)$$

with a  $\chi^2$  of 2.6 for 14 degrees of freedom. The resulting uncertainty in the ratio  $B^2/(AC)$  is large, namely  $B^2/(AC) = 2.3_{-1.0}^{+1.7}$ , signaling the need for much greater precision in order to test our Eq. (48) in a model-independent way.

However, we can attempt to test our predictions by restricting the fit to the form

$$J = \alpha \phi_B^3 \ln(\phi_B^2/v_B^2) + \beta v_B^2 \phi_B (1 - \phi_B^2/v_B^2) \quad (A4)$$

(see Eq. (46)). This gives  $\alpha = 0.065 \pm 0.015$ ,  $\beta = -0.743 \pm 0.028$  and the  $\chi^2$  is again 2.6 for 15 degrees of freedom. Fixing  $\alpha = 0$ , corresponding to a potential of classical form without a  $\phi^4 \ln \phi^2$  term, would give a much poorer fit ( $\chi^2 = 22.5$  for 16 degrees of freedom). The ratio of derivatives  $B^2/(AC)$  is  $2.74 \pm 0.06$ , which corresponds, in our terms, to a substantial and negative  $m_0^2$  (i.e., to a small  $\zeta = 0.08 \pm 0.02$ ) well past the phase transition and also well past the CSI situation  $\zeta = 1$ . By comparing the fitted  $\alpha$  parameter with the corresponding coefficient in (46), we find the constant-field rescaling factor to be

$$Z_\phi = 1.99 \pm 0.25. \quad (A5)$$

Hence, the corresponding renormalized vacuum value is

$$v \equiv v_R = \frac{v_B}{Z_\phi^{1/2}} = 0.31 \pm 0.02. \quad (A6)$$

From these numbers we can evaluate the  $m_0^2/(8\pi^2 I_{-1})$  term in  $m_B^2$  (Eq. (34)). Using Eqs. (38), (42), this is  $6\pi^2 v^2 \zeta(\zeta - 1)/Z_\phi \approx -0.21$ . This agrees well the fact that at this  $m_B^2$  of  $-0.6$  we are past the phase-transition value of about  $-0.4$  by an amount  $-0.2$ .

The physical mass in the broken phase (see Eq. (43)) is

$$m_h = 0.78 \pm 0.05. \quad (A7)$$

It is clear that we are very far from the continuum limit which, in our approach, should exhibit an exponentially small mass gap in lattice units.

One may observe that our values for  $m_h$  and  $Z_\phi$  do not agree with the corresponding quantities quoted by HMP [16, 17, 40]:

$$m_h^{HMP} \sim 0.53 \pm 0.02, \quad (A8)$$

$$Z^{HMP} \sim 0.83 \pm 0.03. \quad (A9)$$

We attribute this discrepancy to the fact that HMP assumed that there was only a single  $Z$ , i.e., that  $Z_\phi = Z_h$ . The point is that from the curve  $J = J(\phi_B)$  alone one cannot disentangle  $m_h$  from  $Z_\phi$  without extra information. In fact,  $m_h$  and  $Z_\phi$  enter Eq. (47) only in the combination

$$A = \frac{m_h^2}{Z_\phi}. \quad (A10)$$

If we compute this ratio for the HMP quantities we find  $0.34 \pm 0.02$ , in very good agreement with the value  $0.31 \pm 0.03$  obtained from our numbers in (A5), (A7). Thus, the discrepancy with HMP has to do with disentangling  $m_h$  from  $Z_\phi$ . HMP did this by computing the shifted-field inverse propagator (which requires the subtraction of disconnected pieces), but we believe that those results are misleading because of the assumption that  $Z_\phi = Z_h = Z$ . In view of this problem we shall stop here and await a new, high-statistics lattice calculation.

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